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Some peculiarities of motion of neutral and charged test particles in the field of a spherically symmetric charged object in General Relativity

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Abstract We propose the method of investigation of radial motions for charged and neutral test particles in the Reissner-Nordström field by means of mass potential. In this context we analyze special features of interaction of charges and their motions in General Relativity and construct the radial motion classification. For test particles and a central source with charges q and Q, respectively, the conditions of attraction (when $qQ > 0$) and repulsion (when $qQ < 0$) are obtained. The conditions of motionless test particle states with respect to the central source are investigated and, in addition, stability conditions for such static equilibrium states are found. It is shown that stable states are possible only for the bound states of weakly charged particles in the field of a naked singularity. Frequencies of small oscillations of test particles near their equilibrium positions are also found.

Keywords Motions classification · Mass potential · Stability conditions

1 Introduction

Investigation of the test particle motion in General Relativity is a classical research method of the structure and properties of space-time near gravitating mass. If a gravitational field source is electrically charged, it substantially influences the space-time geometry. Respectively, the motion character of test bodies is changed. Therefore, the investigation of motion peculiarities for charged particles in such spaces is of a great physical interest as a part of general research of charged configurations in General Relativity.

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In this paper we consider the test particles moving radially in the gravitational field of a spherically symmetric source carrying mass M and charge Q. This field is described by Reissner-Nordström metric

$$ds^2 = Fc^2dT^2 - F^{-1}dR^2 - R^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where

$$F = 1 - \frac{2\gamma M}{c^2R} + \frac{\gamma Q^2}{c^4R^2}\,, \tag{2}$$

γ and c are the gravitational constant and the velocity of light respectively. The charge Q also generates an electric field with potential

$$\varphi = \frac{Q}{R}\,. \tag{3}$$

Note that one can differentiate the following 3 types of the spherically symmetric charged relativistic objects depending on the relation between their masses and charges: a charged black hole (BH) ($\sqrt{\gamma}M > |Q|$), an extremely charged black hole (EBH) ($\sqrt{\gamma}M = |Q|$), a the super-extremely charged object ($\sqrt{\gamma}M < |Q|$ — naked singularity).

The standard qualitative analysis method for particle dynamics uses the velocity potential U_V. This potential results from the radial equation of particle motion $\left(mc^2dR/ds\right)^2 = -U_V(M,Q,m,q,E,L,R)$ (see, for example, [1,2,3,4,6,7]) where parameters m and q are particle mass and charge, E and L are its total energy and angular orbital momentum in the Reissner-Nordström field with parameters M and Q. The inequality $U_V(M,Q,m,q,E,L,R) \leq 0$ defines the regions of admissible radial motions of the particle. The solutions of the equation $U_V(M,Q,m,q,E,L,R) = 0$ with respect to R specifies the turning points. Of course, this method is effective for a complete system of the first integrals of the motion equations. In this case the radial motion equation of the first order is possible.

In order to decrease the number of parameters $\{M,Q,m,q,E,L\}$ and simplify the equations one can use a scale invariance of the dynamical system (if such invariance is present) and the effective potential method additionally. This potential is defined as a solution of the equation $U_V(M,Q,m,q,E,L,R) = 0$ with respect to one of parameters $\{M,Q,m,q,E,L\}$. The solution is interpreted as potential corresponding to the given parameter. The parameter choice depends on the statement of the problem. Thus, the mass potential $U_M = U_M(Q,m,q,E,L,R)$ is determined as a solution of equation $U_V(U_M,Q,m,q,E,L,R) = 0$ with respect to U_M, the energy potential $U_E = U_E(M,Q,m,q,L,R)$ is found as a solution of equation $U_V(M,Q,m,q,U_E,L,R) = 0$ with respect to U_E, and the charge potential $U_q = U_q(M,Q,m,E,L,R)$ is determined as a solution of equation $U_V(M,Q,m,U_q,E,L,R) = 0$ with respect to U_q, etc. It is easy to see that turning points in the above cases are defined by relations $U_M = M$, $U_E = E$, and $U_q = q$ respectively. Notice that the study of the radial motion of charged test particles in the Reissner-Nordström field by means of energy potential (U_E) was carried out in [8]. In fact, the methods of charged (U_q) and mass (U_M) potentials were used in [4] and [5], respectively.

It turns out that the parameters included into the velocity potential linearly lead to the simplest potentials. In this case, the admissible motion region is defined by one inequality for a new potential. For example, it is more convenient to use the mass potential $U_M = U_M(Q,m,q,E,R)$, studying the one-dimensional radial motion of the

neutral and charged test particles ($L = 0$) in the field of a spherically symmetric charged source. This potential application makes it possible to construct an evident motion picture and gives a simple and natural classification of particle trajectories.

The radial trajectories of particles in the Reissner-Nordström field were studied in [4,8,9,10]. One can also find the detailed motion description of neutral and charged particles in the Reissner-Nordström field in [11]. The special motion feature in the field of a charged source is sometimes the fact that both neutral and charged particles with $qQ < 0$ repel from the gravitating center [12,13,14,15]. In general, it takes place in the case of charged shells and other matter distributions. This phenomenon is directly associated with the stability of charged relativistic configurations against a gravitational contraction and with conditions of the gravitational collapse prevention.

Therefore, one of the main aims of such research is to obtain the conditions when falling into the center is impossible (equilibrium conditions for charged particles) and to find stability conditions. For example, the paper [16] is devoted to such a problem. However, the mentioned problem was not considered there with the energy conservation law for a particle and so was not solved completely. Stable, neutral and unstable equilibrium conditions for charged particles in the field of a gravitating center were discussed in [4]. In our opinion, those conditions where not obtained in the closed form because they contain not only parameters of the particles and the central source, but also the distance between them.

In the present paper, unlike the above mentioned ones, we give the complete classification of the radial motions of test particles in spherically symmetric gravitational and electric fields of the central source. This classification naturally follows the energy conservation law of particles and results from the mass potential method and the method of the function of horizons developed here for the first time. This paper is organized the following way: section 2 contains the basic relations and the motion equation of charged particles in the Reissner-Nordström field. In section 3 we analyze the special features of the test particle motion and consider the cases of their exotic motion when particles with charge $qQ > 0$ attract to the center while those with charge $qQ < 0$ repel from it. In particular, here we consider the radial oscillations of weakly charged particles in the field of a super-extremely charged object and obtain their oscillation period in terms of proper time. Section 4 deals with stability conditions. Here we find equilibrium conditions for particles in a closed form and obtain both such state stability conditions and equilibrium radius. These results are expressed only in terms of parameters of particles and the gravitating center (M, Q, m, q, E).

In this paper we neglect the radiation of charged particles and the back reaction of radiation. Note that the constructed classification and stability conditions for stable states are valid also in the case of test charged dust spherical shells, for which the problem of radiation and its back reaction vanishes.

2 Test particle motion equation

We take the action for a charged test particle with mass m and charge q in the Reissner-Nordström field in the form

$$S = -\int (mcds + q\varphi dT) = \int \mathcal{L} dT \tag{4}$$

where $\mathcal{L}$ is the Lagrangian of the particle. In the case of radial motions we have

$$\mathcal{L} = -mc\sqrt{Fc^2 - F^{-1}\dot{R}^2} - qQ/R\,. \tag{5}$$

Here a dot denotes the derivative with respect to time T. The action (4) can be obtained, for example, through generalizing an action of special relativity theory for a charged particle in an electromagnetic field [17] into the corresponding action in General Relativity.

The total energy conservation law for a charged particle

$$E = \frac{mc^2 F}{\sqrt{F - F^{-1}\dot{R}^2/c^2}} + \frac{qQ}{R} = mc^3 F\,\frac{dT}{ds} + \frac{qQ}{R} = const \tag{6}$$

implies the world line equation

$$\left(mc^2\frac{dR}{ds}\right)^2 = \left(E - \frac{qQ}{R}\right)^2 - m^2c^4\left(1 - \frac{2\gamma M}{c^2 R} + \frac{\gamma Q^2}{c^4 R^2}\right) \equiv -U_V(M,Q,m,q,E,R)\,, \tag{7}$$

$$mc^2\frac{dT}{ds} = \frac{1}{cF}\left(E - \frac{qQ}{R}\right)\,. \tag{8}$$

Similar equations were obtained in [4,6,8,10]. Here

$$U_V(M,Q,m,q,E,R) = m^2c^4 - E^2 - (\gamma m^2c^2 M - EqQ)\frac{2}{R} + (\gamma m^2 - q^2)\frac{Q^2}{R^2} \tag{9}$$

is the velocity potential. Admissible motions are determined by the inequality $U_V \le 0$. The solutions of the equation $U_V(M,Q,m,q,E,R) = 0$ with respect to R define the turning points. Using the equation (7) for the particle acceleration we find

$$\frac{d^2R}{ds^2} = \frac{1}{m^2c^4}\left[\left(EqQ - \gamma m^2c^2M\right)\frac{1}{R^2} + \left(\gamma m^2 - q^2\right)\frac{Q^2}{R^3}\right]\,. \tag{10}$$

The given relations yield the trajectory classification of radially moving particles. Further, one can find static equilibrium positions of particles and examine their stability.

3 Radial motion classification

The object of our classification is a central source with parameters $\{M, Q\}$ and a test particle with parameters $\{m, q, E\}$ moving in the field of the source. It is inconvenient to classify motions by means of the velocity potential $U_V(M,Q,m,q,E,R)$ depending on all five parameters $\{M,Q,m,q,E,R\}$. Note that the Lagrangian (5), energy (6), and the motion equation (7) are invariant with respect to scale transformation

$$(M,Q,m,q,E,R,T,s) \longrightarrow (aM,aQ,am,aq,aE,aR,aT,as)\,. \tag{11}$$

Therefore, we fix one of the parameters, for example, $Q \neq 0$, and, to be definite, assume that $Q > 0$ and $M > 0$.

The subsequent simplification and decrease of the number of parameters is achieved by introducing a new potential. Since the parameter M is included into the velocity potential (9) linearly, it is convenient to use the mass potential U_M defined by

$$U_V(U_M,Q,m,q,E,R) = m^2c^4 - E^2 - (\gamma m^2c^2U_M - EqQ)\frac{2}{R} + (\gamma m^2 - q^2)\frac{Q^2}{R^2} = 0. \tag{12}$$

Hence, for the mass potential we obtain

$$U_M = \frac{1}{2\gamma m^2 c^2}\left[(m^2c^4 - E^2)R + 2EqQ + \left(\gamma m^2 - q^2\right)\frac{Q^2}{R}\right]. \tag{13}$$

It follows from (9) that

$$U_M - M = \frac{R}{2\gamma m^2 c^2}\, U_V\,. \tag{14}$$

The condition $U_V \leq 0$ gives the inequality $U_M(Q,m,q,E,R) \leq M$ defining the admissible regions of particle motions in terms of the mass potential. The equation solutions $U_M(Q,m,q,E,R) = M$ with respect to R specify the turning points.

Note that the potential U_M depends only on the particle parameters and has the form of a linear-fractional function, in addition. Therefore, the mass potential properties are determined by its asymptotical behavior at $R \to 0$ and at $R \to \infty$, i.e. by coefficients at R and at $1/R$.

It turns out that the mass potential behavior U_M when $R \to 0$ depends on the electrical characteristics of the particle:

1) $U_M \to +\infty$ if $\gamma m^2 > q^2$ (the weakly charged particle),
2) $U_M \to EqQ/\gamma m^2 c^2$, if $\gamma m^2 = q^2$ (the extremely charged particle),
3) $U_M \to -\infty$ if $\gamma m^2 < q^2$ (the super-extremely charged particle);

while the behavior of U_M when $R \to \infty$ depends only on the energy characteristics of the particle:

1) $U_M \to +\infty$ if $E^2 < m^2c^4$ (the bound states of the particle),
2) $U_M \to EqQ/\gamma m^2 c^2$, if $E^2 = m^2c^4$ (the particle with a critical mass),
3) $U_M \to -\infty$, if $E^2 > m^2c^4$ (the unbound states of the particle).

In coordinates $\{R, U\}$ the plot of the mass potential at the fixed parameters $\{Q, m, q, E\}$ is represented by a curve $U = U_M(Q,m,q,E,R)$. The regions of admissible motions are determined by the horizontal segments of straight lines $U = M = const$ lying above the curve $U = U_M(Q,m,q,E,R)$. Intersection points of the curve $U = U_M(Q,m,q,E,R)$ and the line $U = M = const$ give the turning radii.

The complete history of particles can be observed on the Penrose diagram for the extended Reissner-Nordström space-time. This problem was studied in [9,10,11,13,18, 19] and, therefore, is not considered here. Nevertheless, we plot the curves of horizons for the Reissner-Nordström metric to have some information about the space-time structure. With this aim in view, we introduce the additional function of horizons $U_h = U_h(Q,R)$ as a solution of the equation $F(U_h,Q,R) = 1 - 2\gamma U_h/c^2R + \gamma Q^2/c^4R^2 = 0$ with respect to U_h:

$$U_h = U_h(Q,R) = \frac{1}{2}\left(\frac{Rc^2}{\gamma} + \frac{Q^2}{Rc^2}\right). \tag{15}$$

If the charge Q is fixed, the function of horizons $U_h(Q,R)$ determines the BH mass with the horizon radius R. It is easy to see that $U_h \geq |Q|/\sqrt{\gamma}$ and we have the minimum $(U_h)_{min} = |Q|/\sqrt{\gamma} = M$ for the EBH at $R = |Q|\sqrt{\gamma}/c^2$. In coordinates $\{R, U\}$, the intersection points of the curve $U = U_h(Q,R)$ and the straight line $U = M = const$ give the radii of horizons $R_\pm$ which the particle passes through.

Comparing functions $U_M(Q,m,q,E,R)$ and $U_h(Q,R)$, we get the relation

$$U_h(Q,R) = U_M(Q,m,q,E,R) + \frac{1}{2\gamma m^2 c^2}\left(E\sqrt{R} - \frac{qQ}{\sqrt{R}}\right)^2. \tag{16}$$

It follows from (16) that

$$U_h(Q,R) \ge U_M(Q,m,q,E,R)\,. \tag{17}$$

This means that in coordinates $\{R,U\}$ the curve $U = U_h(Q,R)$ lies always above the curve $U = U_M(Q,m,q,E,R)$. The intersection points of the straight line $U = M = const$ with horizons curve $U = U_h(Q,R)$ are in the region of admissible motions $U_M \le M$. The latter corresponds to the case when the turning radii cannot be in T-region ($R_- < R < R_+$ where $R_\pm = c^{-2}(\gamma M \pm \sqrt{\gamma^2 M^2 - \gamma Q^2})$ are exterior and interior Reissner-Nordström horizons) where the radial coordinate becomes time-like. For neutral particles with $E = 0$ the potential $U_M(Q,m,q,E,R)$ coincides with the function $U_h(Q,R)$. E The equality $U_h(Q,R) = U_M(Q,m,q,E,R)$ at $E \ne 0$, $q \ne 0$ defines a tangency point of these curves: $R_t = qQ/E$. The same point for a particle with energy $E = qQ/R_+$ is a turning point on event horizon [11] with $R_t = R_+$. Thus, the particle never passes into R-regions ($R < R_-$ and $R_+ < R$). If function U_M has a minimum, then we have $(U_M)_{min} \le (U_h)_{min} = |Q|/\sqrt{\gamma}$ from the inequality (17).

Let us consider the particle acceleration. It follows from (10) that neutral particle acceleration vanishes at $R = Q^2/Mc^2$. Thus, in the region $R > Q^2/Mc^2$ attraction takes place, whereas repulsion occurs in the region $R < Q^2/Mc^2$.

For positively charged particles the acceleration vanishes at

$$R = R_a \equiv \frac{(\gamma m^2 - q^2)Q^2}{\gamma m^2 c^2 M - EqQ}\,. \tag{18}$$

From the condition $R_a > 0$ and (10) it is clear that attraction takes place at $R > R_a$ for weakly charged particles with $0 < q < \sqrt{\gamma} m$ and $\gamma m^2 c^2 M > EqQ$. Attraction is observed also at $0 < R < R_a$ for super-extremely charged particles with $q > \sqrt{\gamma} m$ and $\gamma m^2 c^2 M < EqQ$. For super-extremely charged particles with $\gamma m^2 c^2 M \ge EqQ$ attraction occurs for all $0 < R < \infty$.

For extremely charged particles the acceleration sign depends only on the central source and particle parameters

$$\frac{d^2R}{ds^2} = \frac{\gamma}{c^4 R^2}\left(E\,\frac{Q}{q} - Mc^2\right)\,. \tag{19}$$

If $E/q < Mc^2/Q$, attraction can take place, but $E/q > Mc^2/Q$ results in repulsion. When the energy-to-charge ratio of a particle E/q is equal to the ratio of the "total energy"-to-charge of the central object Mc^2/Q (i.e. $E/q = Mc^2/Q$), the gravitation and electric interactions are completely compensated and $d^2R/ds^2 = 0$. Extremely charged particles with a critical mass also move with the vanishing acceleration in the field of an EBH.

For negatively charged particles repulsion is possible only for weakly charged particles $-\sqrt{\gamma} m < q < 0$ in the region

$$0 < R < \tilde{R}_a \equiv \frac{(\gamma m^2 - q^2)Q^2}{\gamma m^2 c^2 M + E|q|Q}\,. \tag{20}$$

Let us introduce the dimensionless quantities

$$\begin{array}{lll} \mathcal{U}_M = \sqrt{\gamma} U_M/Q\,, & \mathcal{U}_h = \sqrt{\gamma} U_h/Q\,, & x = Rc^2/\sqrt{\gamma} Q\,, \\ \varepsilon = E/mc^2\,, & \beta = q/\sqrt{\gamma} m\,, & \mathcal{M} = \sqrt{\gamma} M/Q\,. \end{array} \tag{21}$$

Then the expressions (13) and (15) can be rewritten as

$$\mathcal{U}_M(\varepsilon,\beta,x) = \frac{1}{2}\left[(1-\varepsilon^2)x + 2\varepsilon\beta + \left(1-\beta^2\right)\frac{1}{x}\right], \tag{22}$$

$$\mathcal{U}_h(x) = \frac{1}{2}\left(x + \frac{1}{x}\right). \tag{23}$$

Function graphs $\mathcal{U}_M(x)$ and $\mathcal{U}_h(x)$ in dimensionless coordinates $\{x\,,\mathcal{U}\}$ are displayed in Fig. 1-3 in solid and dashed lines, respectively.

It is possible to specify basic particle motion types depending on the relations between parameters $\{m, q, E\}$ and the central source mass M:

1. Weakly charged particles: $\gamma m^2 > q^2$ or $\beta^2 < 1$ (Fig. 1). We can differentiate several motion cases with regard to energy parameters.

1.1. Bound states of weakly charged particles: $E^2 < m^2c^4$ or $\varepsilon^2 < 1$. The particles move inside the potential well between the turning radii R_1 and R_2 (Fig. 1(a)) where

$$R_{1,2} = \frac{\gamma M m^2c^2 - EqQ \mp m\sqrt{\triangle}}{m^2c^4 - E^2}, \tag{24}$$

here

$$\triangle = \gamma(EQ - qMc^2)^2 + c^4(\gamma M^2 - Q^2)(\gamma m^2 - q^2). \tag{25}$$

Hence one can see that $\triangle > 0$ when $Q \le \sqrt{\gamma}M$. If $Q > \sqrt{\gamma}M$, the requirement $\triangle > 0$ gives the energy condition either $E \ge E_+$ or $E \le E_-$ for the particle where

$$E_\pm = \frac{c^2}{\sqrt{\gamma}}\left(\frac{q}{Q}M\sqrt{\gamma} \pm \sqrt{\gamma m^2 - q^2}\sqrt{1 - \frac{\gamma M^2}{Q^2}}\right). \tag{26}$$

The central source mass is bounded by the inequality $(U_M)_{min} \le M < \infty$ where

$$(U_M)_{min} = U_M(R_{extr}) = \frac{Q}{\gamma m^2c^2}\left(Eq + \sqrt{(m^2c^4 - E^2)(\gamma m^2 - q^2)}\right) < \frac{Q}{\sqrt{\gamma}}. \tag{27}$$

If $M = (U_M)_{min}$, the particle is at the bottom of the potential well. Thus, relations $dR/ds = 0$ and $d^2R/ds^2 = 0$ are satisfied, and the particle with energy $E = E_+$ remains motionless at the distance

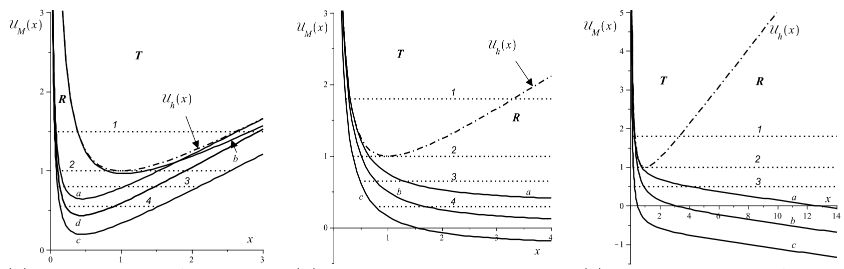
$$R_{extr} = Q\sqrt{\frac{\gamma m^2 - q^2}{m^2c^4 - E_+^2}} \tag{28}$$

from the super-extremely charged central source. Taking into account (26), this formula can be rewritten as

$$R_{extr} = \frac{\sqrt{\gamma}Q^2}{c^2}\,\frac{\sqrt{\gamma m^2 - q^2}}{\sqrt{\gamma}M\sqrt{\gamma m^2 - q^2} - qQ\sqrt{1 - \gamma M^2/Q^2}}. \tag{29}$$

If $M > (U_M)_{min}$, the equation (7) yields the motion trajectory $R = R(s)$ in an implicit form

$$\begin{aligned} s(R) - s_0 = {} & \frac{2mc^2(\gamma M m^2c^2 - EqQ)}{(m^2c^4 - E^2)^{3/2}}\arctan\sqrt{\frac{R - R_1}{R_2 - R}} - \\ & - \frac{mc^2}{m^2c^4 - E^2}\sqrt{(E^2 - m^2c^4)R^2 + 2(\gamma M m^2c^2 - EqQ)R - Q^2(\gamma m^2 - q^2)}\,. \end{aligned} \tag{30}$$

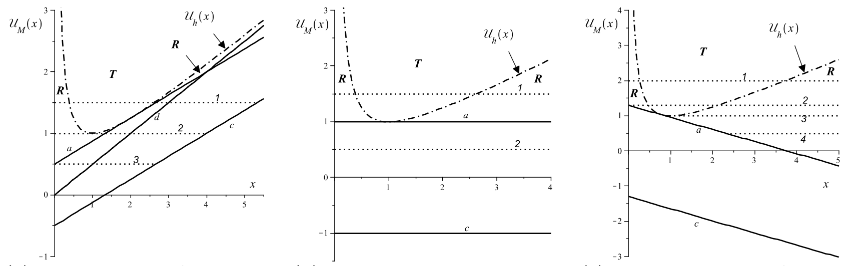


(a) Bound states of particles ($E^2 < m^2c^4$).

(b) Particles with a critical mass ($E^2 = m^2c^4$).

(c) Unbound states of particles ($E^2 > m^2c^4$).

Fig. 1 Weakly charged particles ($\gamma m^2 > q^2$).

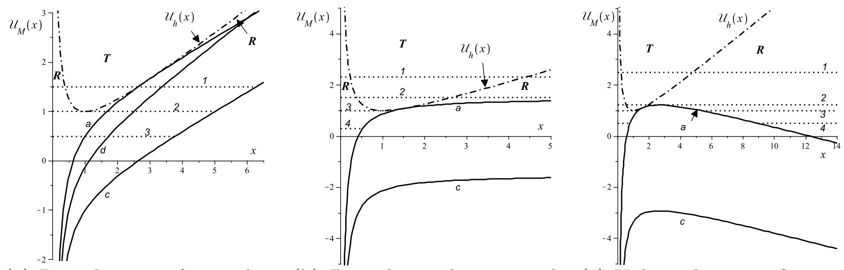


(a) Bound states of particles ($E^2 < m^2c^4$).

(b) Particles with a critical mass ($E^2 = m^2c^4$).

(c) Unbound states of particles ($E^2 > m^2c^4$).

Fig. 2 Extremely charged particles ($\gamma m^2 = q^2$).



(a) Bound states of particles ($E^2 < m^2c^4$).

(b) Particles with a critical mass ($E^2 = m^2c^4$).

(c) Unbound states of particles ($E^2 > m^2c^4$).

Fig. 3 Super-extremely charged particles ($\gamma m^2 < q^2$).

In figures 1, 2 and 3 the admissible motion regions for given $\mathcal{M}$ are determined by the horizontal segments of dotted lines $\mathcal{U} = \mathcal{M} = const$ lying above the curve $\mathcal{U} = \mathcal{U}_M(\varepsilon, \beta, x)$. The segments of lines $\mathcal{U} = \mathcal{M} > 1$, $\mathcal{U} = \mathcal{M} = 1$ and $\mathcal{U} = \mathcal{M} < 1$ describe particle motions in the field of a BH, EBH, and super-extremely charged object (naked singularity), respectively. The intersection points of the mass potential curve $\mathcal{U} = \mathcal{U}_M(\varepsilon, \beta, x)$ and straight line $\mathcal{U} = \mathcal{M} = const$ give the turning radii. The curves a, b and c correspond to mass potentials for the cases $q > 0$, $q = 0$ and $q < 0$, respectively. The curve d corresponds to the mass potential of a particle with energy $E = 0$.

In this case the particle oscillates anharmonically with the period

$$T = \frac{2}{c}(s(R_2) - s(R_1)) = 2\pi mc \frac{\gamma M m^2 c^2 - EqQ}{(m^2c^4 - E^2)^{3/2}} \tag{31}$$

with respect to the proper time. Thus, if $Q > \sqrt{\gamma}M$, then the particle oscillates near the equilibrium position R_{extr} in the field of the super-extremely charged object. In the case of small deviations of the charged particle from the equilibrium position we have harmonic oscillations with the period

$$T = \frac{2\pi\gamma m Q^3}{c^3} \frac{\sqrt{\gamma m^2 - q^2}}{\left(M\sqrt{\gamma}\sqrt{\gamma m^2 - q^2} - q\sqrt{Q^2 - \gamma M^2}\right)^2}. \tag{32}$$

If $Q < \sqrt{\gamma}M$, the region II ($R_- \leq R \leq R_+$) of the Penrose diagram for the maximally extended Reissner-Nordström solution enters the admissible motion region ($R_1 \leq R \leq R_2$). The particle trajectory starting in the given region I crosses the horizon R_+, enters the region II, and then crosses the horizon R_-. After that, the particle reaches the turning point $R = R_1$ in the region III, returns back, and passes through a new region II into another asymptotically flat region I of the Penrose diagram (see, for example, [20], Fig. 25). Here the particle reaches the turning point R_2 and comes back to the horizon R_+ again, etc. Therefore, the particle moves along the infinite chain of regions I, II, III and horizons $R_\pm$: ... $\to$ I $\to R_+ \to$ II $\to R_- \to$ III $\to R_- \to$ II $\to R_+ \to$ I $\to$ ....

In the case of EBH ($Q = \sqrt{\gamma}M$) the particle moves along the infinite chain of regions I, III and horizons $R_e = \gamma M/c^2$: ... $\to$ I $\to R_e \to$ III $\to R_e \to$ I $\to$ ... (see [20], Fig. 26).

The bound state of a neutral particle with $E^2 < m^2c^4$ and $q = 0$ is a special case of the bound state of a weakly charged particle. The particle is in the potential well (Fig. 1(a), curve b) and moves within the region

$$R_{01} = \frac{\gamma M m^2 c^2 - m\sqrt{\triangle}}{m^2c^4 - E^2} \leq R \leq R_{02} = \frac{\gamma M m^2 c^2 + m\sqrt{\triangle}}{m^2c^4 - E^2} \tag{33}$$

where $\triangle = \gamma^2 M^2 m^2 c^4 - \gamma Q^2(m^2c^4 - E^2)$. The central source mass varies in the interval $(U_M)_{min} \leq M < \infty$ where

$$(U_M)_{min} = U_M(R_{extr}) = \frac{Q}{\sqrt{\gamma}}\sqrt{1 - \frac{E^2}{mc^2}} < \frac{Q}{\sqrt{\gamma}}\,. \tag{34}$$

In the case when $M = (U_M)_{min}$ the neutral particle with energy

$$E = mc^2\sqrt{1 - \gamma M^2/Q^2} \tag{35}$$

is at the bottom of the potential well and remains motionless at the distance

$$R_{extr0} = \frac{\sqrt{\gamma}\, mQ}{\sqrt{m^2c^4 - E^2}} = \frac{Q^2}{Mc^2} \tag{36}$$

from the super-extremely charged object. In the case when $M > (U_M)_{min}$ the particle oscillates anharmonically and we have

$$s(R) - s_0 = \frac{2\gamma m^3 c^4 M}{(m^2c^4 - E^2)^{3/2}} \arctan\sqrt{\frac{R - R_{01}}{R_{02} - R}} - mc^2\sqrt{\frac{(R - R_{01})(R_{02} - R)}{m^2c^4 - E^2}} \tag{37}$$

with the period $T = 2\pi\gamma m^3 c^3 M/(m^2c^4 - E^2)^{3/2}$ with respect to the proper time.

If $Q > \sqrt{\gamma}M$, the particle oscillates near the point R_{extr0} in the naked singularity field. In the case of small deviations of the neutral particle from the equilibrium position R_{extr0} we have harmonic oscillations with the period

$$T = \frac{2\pi Q^3}{M^2 c^3 \sqrt{\gamma}} \tag{38}$$

which is particle mass independent, as we see.

If $Q < \sqrt{\gamma}M$, the particle starting in a given region I passes through regions II, III and II and appears in another asymptotically flat region I on the Penrose diagram (see [20], Fig. 25).

In the case when $q = 0$ and $E = 0$ functions $U_h(Q,R)$ and $U_M(Q,m,q,E,R)$ coincide that results in equality $R_{01,02} = R_\pm$ and the particle moves along the world line

$$s(R) - s_0 = 2\gamma M/c^2 \arctan\sqrt{\frac{R-R_-}{R_+-R}} - R\sqrt{-F}\,,$$

traveling along the infinite chain of regions II, sequentially arriving to horizons R_+ and R_-: ... $\to$ II $\to R_- \to$ II $\to R_+ \to$ II $\to$ ... (see. [20], Fig. 25). Thus the particle moves from a horizon R_+ to the nearest following horizon R_+ during the proper time $T = 2\pi\gamma M/c^3$.

1.2. Weakly charged particles with a critical mass: $E^2 = m^2c^4$ or $\varepsilon^2 = 1$ (Fig. 1(b)). The mass potential has a simple form

$$U_M(Q,m,q,R) = \frac{1}{2\gamma m^2 c^2}\left[2mqQc^2 + \left(\gamma m^2 - q^2\right)\frac{Q^2}{R}\right]. \tag{39}$$

The central object mass is bounded by inequality $qQ/\gamma m \le M < \infty$. For all masses $M > qQ/\gamma m$ the motion region is bounded by the condition

$$R \ge R_3 = \frac{Q^2(\gamma m^2 - q^2)}{2mc^2(\gamma m M - qQ)}\,. \tag{40}$$

The particles start falling from the infinity with zero initial velocity, reach the turning radius $R = R_3$, and eventually come back to infinity again. In the case when $\gamma m M = qQ$ the particles are at rest at $R = +\infty$.

For a neutral particle with a critical mass the motion region is bounded by the condition $R \ge R_{03} = Q^2/2Mc^2$.

1.3. Unbound states of weakly charged particles: $E^2 > m^2c^4$ or $\varepsilon^2 > 1$ (Fig. 1(c)). In this case motion occurs in the region

$$R \ge R_4 = \frac{EqQ - \gamma M m^2 c^2 + m\sqrt{\triangle}}{E^2 - m^2c^4}\,. \tag{41}$$

The particles start moving from the infinity with the initial velocity $dR/ds = -\sqrt{E^2/m^2c^4 - 1}$. Then they reach the turning radius $R = R_4$ and go back to the infinity again.

For neutral particles in unbound states the motion region is bounded by the condition $R \ge R_{04}$ where

$$R_{04} = \frac{-\gamma M m^2 c^2 + m\sqrt{\gamma^2 M^2 m^2 c^4 + \gamma Q^2(E^2 - m^2c^4)}}{E^2 - m^2c^4}\,. \tag{42}$$

2. Extremely charged particles: $\gamma m^2 = q^2$ or $\beta^2 = 1$ (Fig.2). In this case the mass potential depends on R linearly

$$U_M(Q,q,E,R) = \frac{1}{2\gamma q^2 c^2}\left[(q^2c^4 - \gamma E^2)R + 2\gamma qQE\right] . \tag{43}$$

2.1. Bound states of extremely charged particles: $E^2 < m^2c^4$ or $\varepsilon^2 < 1$ (Fig.2(a)). The central object mass is bounded by the inequality $M \geq EQ/\sqrt{\gamma}mc^2$. If $M > EQ/\sqrt{\gamma}mc^2$, the particles move within the region $0 \leq R \leq R_5$ (see dashed lines 1, 2 and 3 for all particles except particles with $q > 0$ for straight line 3). Here

$$R_5 = \frac{2\sqrt{\gamma}m(\sqrt{\gamma}mMc^2 + EQ)}{m^2c^4 - E^2} . \tag{44}$$

In the case $M = EQ/\sqrt{\gamma}mc^2$ the particle will always stay at singularity $R = 0$

2.2. Extremely charged particles with a critical mass: $E^2 = m^2c^4$ or $\varepsilon^2 = 1$ (Fig. 2(b)). The mass potential is independent of the radius $U_M(Q,m,q) = mQ/q$. We have for the velocity and acceleration of particles

$$\left(\frac{dR}{ds}\right)^2 = \left(M - \frac{q}{|q|}\frac{Q}{\sqrt{\gamma}}\right)\frac{2\gamma}{Rc^2} , \qquad \frac{d^2R}{ds^2} = -\left(M - \frac{q}{|q|}\frac{Q}{\sqrt{\gamma}}\right)\frac{\gamma}{R^2c^2} < 0$$

The central source mass is bounded by the inequality $M \geq qQ/|q|\sqrt{\gamma}$. In the case when $q > 0$ and $M > Q/\sqrt{\gamma}$ (BH) or $q < 0$ and $M > 0$ particles move in the region $0 \leq R \leq \infty$. Here attraction takes place. If $Q = M\sqrt{\gamma}$ (EBH), both the velocity and the acceleration of particles with $q > 0$ equal zero. The particle are in the neutral equilibrium state and are at rest at arbitrary R.

2.3. Unbound states of extremely charged particles: $E^2 > m^2c^4$ or $\varepsilon^2 > 1$ (Fig. 2(c)). In the case when particles have a charge $q > 0$, there are two possibilities. If the central source mass varies within the interval $0 < M < EQ/\sqrt{\gamma}mc^2$, the particle acceleration is positive $d^2R/ds^2 > 0$ and repulsion takes place. The motion occurs in the region $R \geq R_6$ (straight lines 3 and 4) where

$$R_6 = \frac{2\sqrt{\gamma}m(EQ - \sqrt{\gamma}Mmc^2)}{E^2 - m^2c^4} . \tag{45}$$

Therefore, a particle coming from the infinity reaches R_6 and goes back to the infinity again. If $M \geq EQ/\sqrt{\gamma}mc^2$, the particle acceleration is negative $d^2R/ds^2 < 0$ and attraction takes place. Particles coming from the infinity reach singularity $R = 0$ (straight lines 1 and 2). In the case $q < 0$ particles fall to singularity for all $M > 0$.

3. Super-extremely charged particles: $\gamma m^2 < q^2$ or $\beta^2 > 1$ (Fig. 3).

3.1. Bound states of super-extremely charged particles: $E^2 < m^2c^4$ or $\varepsilon^2 < 1$. Particles move in the bounded region $0 \leq R \leq R_7$ for all $M > 0$ (Fig. 3(a), straight lines 1, 2 and 3) where

$$R_7 = \frac{\gamma Mm^2c^2 - EqQ + m\sqrt{\triangle}}{m^2c^4 - E^2} . \tag{46}$$

3.2. Super-extremely charged particles with a critical mass: $E^2 = m^2c^4$ or $\varepsilon^2 = 1$ (Fig. 3(b)). The mass potential has the form (39). If the central source mass is in

the interval $0 < M < qQ/\gamma m$, particles always move within the region $0 \le R \le R_8$ (straight lines 3 and 4) where

$$R_8 = \frac{(q^2 - \gamma m^2)Q^2}{2mc^2(qQ - \gamma Mm)} \,. \tag{47}$$

If $M \ge qQ/\gamma m$, all particles falling from the infinity reach singularity $R = 0$ (straight lines 1 and 2).

3.3. Unbound states of super-extremely charged particles: $E^2 > m^2c^4$ or $\varepsilon^2 > 1$ (Fig. 3(c)). The mass potential is bounded above $U_M(Q,R) \le (U_M)_{max}$ where

$$(U_M)_{max} = U_M(Q, \tilde{R}_{extr}) = \frac{Q}{\gamma m^2 c^2}\left(Eq - \sqrt{(E^2 - m^2c^4)(q^2 - \gamma m^2)}\right) \,, \tag{48}$$

$$\tilde{R}_{extr} = Q\sqrt{\frac{q^2 - \gamma m^2}{E^2 - m^2c^4}} \,. \tag{49}$$

If $M > (U_M)_{max}$ (straight line 1), a particle with $q > 0$ coming from the infinity inevitably falls to singularity. If $M = (U_M)_{max}$ (straight line 2), a particle falling from the infinity reaches R_{extr}. At this point the conditions $dR/ds = 0$ and $d^2R/ds^2 = 0$ are satisfied and the maximum of the potential corresponds to the unstable equilibrium position. If $M < (U_M)_{max}$ (straight line 3 and 4), the particle motion occurs either in the region $0 \le R \le R_9$, or $R \ge R_{10}$, where the particle reaches singularity. Here

$$R_{9,10} = \frac{EqQ - \gamma M m^2 c^2 \mp m\sqrt{\triangle}}{E^2 - m^2c^4} \,. \tag{50}$$

A particle with $q < 0$ traveling from the infinity reaches singularity $R = 0$ for all $M > 0$.

4 Stability conditions

Stationary equilibrium states of a particle at some $R = R_{extr}$ are defined by conditions $dR/ds = 0$ $\,d^2R/ds^2 = 0$. Furthermore, if $d^2R/ds^2 > 0$ at $R < R_{extr}$ and $d^2R/ds^2 < 0$ at $R > R_{extr}$, there exist a stable position R_{extr}. It follows from definitions for the velocity and mass potentials (14) that at the potential well bottom (Fig. 1(a)) in the cases $U(R_{extr}) = 0$ or $U_M(R_{extr}) = M$ the conditions $(\partial U/\partial R)|_{R_{extr}} = 0$ and $(\partial U_M/\partial R)|_{R_{extr}} = 0$ coincide and give a static equilibrium radius R_{extr} (28).

Taking into account (7) and (10), we can eliminate a variable R from the equations $dR/ds = 0$ and $d^2R/ds^2 = 0$. As a result we obtain the equation $\triangle = 0$ (see (25)) which can be rewritten as

$$(m^2c^4 - E^2)(\gamma m^2 - q^2)Q^2 = (m^2c^2\gamma M - EqQ)^2 \,. \tag{51}$$

From here one can find two systems of inequalities

$$|q| < m\sqrt{\gamma}, \quad |E| < mc^2 \tag{52}$$

or

$$|q| > m\sqrt{\gamma}, \quad |E| > mc^2 \,. \tag{53}$$

Rewriting (51) in the form

$$(Q^2 - \gamma M^2)(\gamma m^2 - q^2) = \gamma \left(\frac{EQ}{c^2} - qM \right)^2 , \tag{54}$$

we obtain in a similar way

$$|q| < m\sqrt{\gamma}, \quad |Q| > M\sqrt{\gamma} \tag{55}$$

or

$$|q| > m\sqrt{\gamma}, \quad |Q| < M\sqrt{\gamma}\,. \tag{56}$$

Therefore, from the motion classification and relations (52) and (55) we find the following conditions: $|E| < mc^2$, $|q| < m\sqrt{\gamma}$ and $|Q| > M\sqrt{\gamma}$. This case corresponds to the stable equilibrium of the bound states of a weakly charged particle in the field of a super-extremely charged object [1]. The relations (53) and (56) yield the following conditions: $|E| > mc^2$, $|q| > m\sqrt{\gamma}$ and $|Q| < M\sqrt{\gamma}$. This case corresponds to the instable equilibrium of the unbound states of a super-extremely charged particle in the field of a charged BH. The stability conditions (55) and (56) have been obtained in [16], but there was not pointed out which of the equilibrium states was stable, and a static equilibrium radius has not been found for the particle.

We find from (54) that a charged test particle at the static position R_{extr} (28) has energy $E = E_+$ (see (26)). In turn, a neutral test particle remains motionless at the distance R_{extr0} (36) and has energy (35).

5 Conclusion

A special feature of the charge interaction in General Relativity is the fact that in the case of test particles with charge $q > 0$ moving radially in the field of a central source with charge $Q > 0$ both for weakly charged particles ($q < m\sqrt{\gamma}$) at $R > R_a$ and for super-extremely charged particles ($q > m\sqrt{\gamma}$) at $0 < R < R_a$ attraction is observed (see (18)). If $\gamma m^2 c^2 M \geq EqQ$, attraction takes place for super-extremely charged particles for all values $0 < R < \infty$.

In the case of extremely charged particles $q = m\sqrt{\gamma}$ the acceleration sign is independent of the distance; it depends only on the parameters of the central source and particles. If $E/q < Mc^2/Q$, attraction can take place, but $E/q > Mc^2/Q$ results in repulsion. If the equality $E/q = Mc^2/Q$ is satisfied, gravitational and electric interactions are balanced completely.

In the case $q < 0$ repulsion is possible only for weakly charged particles $-m\sqrt{\gamma} < q < 0$ for radius values $0 < R < \tilde{R}_a$ (see formula (20)).

It is interesting to note that for neutral particles in the Reissner-Nordström field attraction takes place when $R > Q^2/Mc^2$ whereas repulsion occurs when $R < Q^2/Mc^2$.

Introducing the mass potential made it possible to construct a simple and evident classification of possible states for the interacting system under consideration: a test particle with parameters $\{m, E, q\}$ and a central source with parameters $\{M, Q\}$. As a result, it has been shown that a stable static position was possible only for bound states ($E^2 < m^2c^4$) of weakly charged particles ($\gamma m^2 > q^2$) moving in the field of a

[1] It is noteworthy that in [21] it is approved otherwise: the stable system is impossible in the case of a super-extremely charged object $|Q| > M\sqrt{\gamma}$.

charged naked singularity ($\gamma M^2 < Q^2$). In this case the central object mass is equal to the mass potential minimum $M = (U_M)_{min}$ (see (27)). Incidentally energy E_+ and static position R_{extr} for a particle are determined by formulae (26) and (28). If $M > (U_M)_{min}$, the particle oscillates near its equilibrium position R_{extr}. Let us note that the motionless state is also observed for neutral particles. In this case a static position R_{extr0} and energy E are defined by formulae (35) and (36).

It is easy to see that the equilibrium equations (51) and (54) are invariant with respect to the scale transformation $M = aM'$, $Q = aQ'$, $E = aE'$, $m = am'$ and $q = aq'$. Hence, according to (28), $R_{extr} = aR'_{extr}$ and we have the similarity law. If we increase the charge and mass of the central source and, correspondingly, the mass, charge and energy of the particle by a factor a, the static state of the particle with new parameters will remain stable, and its static position radius will be also increased by the factor a.

The obtained stability conditions for the static state of a particle can be generalized both for the case of static states of charged dust shells [22] and the case of charged dust layers in spherically symmetric configurations in General Relativity [23]. These conditions make it possible to construct classical particle-like models of stable static charged dust balls in General Relativity [24].

The existence of the stable lowest state for a neutral or weakly charged particle in the field of a charged naked singularity points out also the existence of stationary quantum states of a charged or neutral particle with a discrete spectrum of energies in the field of a super-extremely charged central source of a Reissner-Nordström type. Thus, according to section 3, in the case of small deviations of a weakly charged particle from the stable positions in the field of a naked singularity harmonic oscillations take place. The oscillation frequency has the form

$$\omega = \frac{2\pi}{T} = \frac{c^3}{\gamma m Q^3} \frac{\left(M\sqrt{\gamma}\sqrt{\gamma m^2 - q^2} - q\sqrt{Q^2 - \gamma M^2}\right)^2}{\sqrt{\gamma m^2 - q^2}} \,. \tag{57}$$

If we formally consider a quantum oscillator corresponding to this case, the lowest state of such system will have energy $E_0 = \hbar\omega/2$. For a neutral particle we will have

$$E_0 = \frac{\hbar}{2} \frac{c^3 M^2 \sqrt{\gamma}}{Q^3} \,. \tag{58}$$

Thus the lowest energy state of a neutral particle depend on the naked singularity parameters M and Q.

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